

A COMPARISON INEQUALITY FOR SUMS OF INDEPENDENT RANDOM VARIABLES

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ABSTRACT. We give a comparison inequality that allows one to estimate the tail probabilities of sums of independent Banach space valued random variables in terms of those of independent *identically distributed* random variables. More precisely, let X_1, \dots, X_n be independent Banach-valued random variables. Let I be a random variable independent of X_1, \dots, X_n and uniformly distributed over $\{1, \dots, n\}$. Put $\tilde{X}_1 = X_I$, and let $\tilde{X}_2, \dots, \tilde{X}_n$ be independent identically distributed copies of \tilde{X}_1 . Then, $P(\|X_1 + \dots + X_n\| \geq \lambda) \leq cP(\|\tilde{X}_1 + \dots + \tilde{X}_n\| \geq \lambda/c)$ for all $\lambda \geq 0$, where c is an absolute constant.

The independent Banach-valued random variables X_1, \dots, X_n are said to *regularly cover* (the distribution of) a random variable Y provided that

$$E[g(Y)] = \frac{1}{n} \sum_{k=1}^n E[g(X_k)],$$

for all Borel functions g for which either side is defined [8]. An easy way of constructing Y , given the independent Banach-valued random variables X_1, \dots, X_n , is to let I be a random variable independent of X_1, \dots, X_n , with values in $\{1, 2, \dots, n\}$ and with each value having equal probability $1/n$, and then put $Y = X_I$. It is easy to see that then X_1, \dots, X_n regularly cover Y . This construction will be useful for our proofs.

If the variables are real valued, then the regular covering condition is easily seen to be equivalent to the condition that the distribution function F of Y is the arithmetic mean of the respective distribution functions F_1, \dots, F_n of X_1, \dots, X_n .

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A variable X' is said to be a *copy* of X if it has the same distribution as X . The main purpose of this paper is then to prove the following result.

Theorem 1. *There exists an absolute constant $c \in (0, \infty)$ such that if X_1, \dots, X_n are independent Banach-valued random variables which regularly cover a random variable \tilde{X}_1 , then:*

$$(1) \quad P(\|X_1 + \dots + X_n\| \geq \lambda) \leq cP(\|\tilde{X}_1 + \dots + \tilde{X}_n\| \geq \lambda/c),$$

for all $\lambda \geq 0$, where $\tilde{X}_2, \dots, \tilde{X}_n$ are independent copies of \tilde{X}_1 .

Remark 1. In the case where the random variables are symmetric, this was shown in [9] (strictly speaking, it was only shown in the real-valued case, but the proof also works for the Banach-valued case).

Remark 2. The inequality converse to (1) is false, even in the special cases of symmetric real random variables. For, suppose that c is an absolute constant such that

$$(2) \quad P(|\tilde{X}_1 + \dots + \tilde{X}_n| \geq \lambda) \leq cP(|X_1 + \dots + X_n| \geq \lambda/c),$$

for all $\lambda \geq 0$, whenever the conditions of Theorem 1 hold with symmetric variables. Fix any $n > \max(1, c)$. Put $X_2 \equiv \dots \equiv X_n \equiv 0$. Let X_1 be such that $P(X_1 = 1) = P(X_1 = -1) = \frac{1}{2}$. Put $\lambda = n$. Then the right hand side of (2) is zero, since $|X_1 + \dots + X_n| \equiv 1$. But the left hand side of (2) is non-zero, since it is easy to see that $P(\tilde{X}_i = 1) = 2^{-n-1}$ for each i (as the \tilde{X}_i are identically distributed, and as \tilde{X}_1 can be taken to be X_I where I is independent of everything else and uniformly distributed on $\{1, \dots, n\}$), so that $P(|\tilde{X}_1 + \dots + \tilde{X}_n| \geq n) \geq (2^{-n-1})^n > 0$.

Remark 3. The main consequence of Theorem 1 is that any upper bound on tail probabilities of sums of independent identically distributed random variables automatically gives a bound on tail probabilities of sums of non-identically distributed independent random variables.

Remark 4. For a very simple application, we give another proof of one side of a result from [8] on randomly sampled Riemann sums. Let $f \in L^2[0, 1]$. For $1 \leq k \leq n$, let x_{nk} be uniformly distributed over $[(k -$

$1)/n, k/n]$, and assume x_{n1}, \dots, x_{nn} are independent for each fixed n . Define the randomly sampled Riemann sum $R_n f = n^{-1} \sum_{k=1}^n f(x_{nk})$. Then the result says that $R_n f$ converges almost surely to the Lebesgue integral $A = \int_0^1 f$. (For a converse in the case where *all* the x_{nk} are independent, not just for fixed n , see [8].) For, by Borel-Cantelli it suffices to show that

$$(3) \quad \sum_{n=1}^{\infty} P(|R_n f - A| \geq \varepsilon) < \infty,$$

for all $\varepsilon > 0$. Let X_1, X_2, \dots be independent identically distributed random variables with the same distribution as f . Note that $f(x_{n1}), \dots, f(x_{nn})$ regularly cover X_1 , and $f(x_{n1}) - A, \dots, f(x_{nn}) - A$ regularly cover $X_1 - A$. Since $f \in L^2$, we have X_1 having a finite second moment, and moreover $E[X_1] = A$, so that by the Hsu-Robbins law of large numbers [6] (see also [3, 4]), we have

$$\sum_{n=1}^{\infty} P(|(X_1 - A) + \dots + (X_n - A)|/n \geq \varepsilon) < \infty,$$

for all $\varepsilon > 0$. By Theorem 1 and the fact that $f(x_{n1}) - A, \dots, f(x_{nn}) - A$ regularly cover $X_1 - A$, we obtain (3).

To prove Theorem 1, we need some definitions and lemmata. If X is a random variable, then let $X^s = X - X'$ be the *symmetrization* of X , where X' is an independent copy of X . We shall always choose symmetrizations so that we have $(X_1 + \dots + X_k)^s = X_1^s + \dots + X_k^s$ whenever we need this identity.

Write $\|X\|_p = (E[\|X\|^p])^{1/p}$, where $\|\cdot\|$ is the norm on the Banach space in which our random variables take values.

Lemma 1. *Let X be a Banach-valued random variable with $\|X\|_2 < \infty$. Then, $\|X\|_2 \leq \|X^s\|_2 + \|E[X]\| \leq 3\|X\|_2$*

Proof. Let X' be an independent copy of X so that $X^s = X - X'$. Let \mathcal{A} be the sigma-algebra generated by X . Then $E[X^s \mid \mathcal{A}] = X - E[X'] = X - E[X]$, and so

$$\|X\|_2 = \|E[X^s + E[X] \mid \mathcal{A}]\|_2 \leq \|X^s + E[X]\|_2 \leq \|X^s\|_2 + \|E[X]\|,$$

where the first inequality used the fact that conditional expectation is a contraction on the Banach-valued L^p spaces, $p \geq 1$ (see, e.g., [2,

Theorem V.1.4]). The rest of the Lemma follows from the triangle inequality. \square

Lemma 2. *Let X_1, \dots, X_n be independent random variables, and let $\tilde{X}_1, \dots, \tilde{X}_n$ be independent identically distributed random variables such that X_1, \dots, X_n regularly cover \tilde{X}_1 . Put $S_n = X_1 + \dots + X_n$ and $\tilde{S}_n = \tilde{X}_1 + \dots + \tilde{X}_n$. Then:*

$$\|S_n\|_2 \leq 12\|\tilde{S}_n\|_2.$$

Proof. Let I_1, \dots, I_n be independent random variables uniformly distributed on the set $\{1, \dots, n\}$. Let $\{X_{i,j}\}_{1 \leq i, j \leq n}$ and $\{X'_{i,j}\}_{1 \leq i, j \leq n}$ be independent arrays of independent random variables, with the arrays independent of the I_i , and such that $X_{i,j}$ and $X'_{i,j}$ both have the same distribution as X_j for all i and j . Without loss of generality we can put $\tilde{X}_i = X_{i,I_i}$. Set $\tilde{X}'_i = X'_{i,I_i}$. Let $\tilde{S}'_n = \tilde{X}'_1 + \dots + \tilde{X}'_n$. Let (X'_1, \dots, X'_n) be an independent copy of (X_1, \dots, X_n) , and put $S'_n = X'_1 + \dots + X'_n$. Observe that $X_1 - X'_1, \dots, X_n - X'_n$ regularly cover $\tilde{X}_i - \tilde{X}'_i$ for all i , and that moreover the $X_i - X'_i$ are symmetric. Thus, by [9, Proposition 1] (which though stated for real valued random variables, holds for the Banach-valued case as well, and with the same proof) we have:

$$(4) \quad \|S_n - S'_n\|_2 \leq 4\|\tilde{S}_n - \tilde{S}'_n\|_2.$$

Also, it is clear that $E[S_n] = E[\tilde{S}_n]$. Combining this with Lemma 1, we see that:

$$\|S_n\|_2 \leq \|S_n - S'_n\|_2 + \|E[S_n]\| \leq 4\|\tilde{S}_n - \tilde{S}'_n\|_2 + 4\|E[\tilde{S}_n]\| \leq 12\|\tilde{S}_n\|_2,$$

as desired. \square

The following Lemma is in effect a special case of a result of Hitzchenko [5].

Lemma 3. *Let X_1, \dots, X_n be independent identically distributed Banach-valued random variables with $\|X_i\| < L$ almost surely for all i . Let $S_k = X_1 + \dots + X_k$. Then:*

$$(E[\|S_n\|])^2 \geq c(E[\|S_n\|^2] - c^{-1}L^2),$$

where $c \in (0, \infty)$ is an absolute constants.

Proof of Lemma 3. By the work of Hitczenko [5], if $S^* = \max_k \|S_k\|$ and $X^* = \max_k \|X_k\|$, then for $q \geq p$:

$$\|S^*\|_q \leq c_0 \frac{q}{p} (\|S^*\|_p + \|X^*\|_q),$$

for a finite absolute constant c_0 . By [7, Corollary 4] we have $\|S^*\|_p \leq c_1 \|S_n\|_p$ for an absolute constant c_1 , as the X_i are identically distributed. The desired inequality easily follows from this with $c = 8c_0^2$ if we let $q = 2$ and $p = 1$. \square

Proof of Theorem 1. Let $I_1, \dots, I_n, \{X_{i,j}\}_{1 \leq i,j \leq n}, \{X'_{i,j}\}_{1 \leq i,j \leq n}, S'_n$ and \tilde{S}'_n be as in the proof of Lemma 2. Applying [9, Proposition 1] (which works for Banach-valued variables as already stated), we see that

$$(5) \quad P(\|S_n - S'_n\| \geq \lambda) \leq 8P(\|\tilde{S}_n - \tilde{S}'_n\| \geq \lambda/2) \leq 16P(\|\tilde{S}_n\| \geq \lambda/4),$$

for all λ , where the second inequality followed from the inequality that $P(\|X^s\| \geq t) \leq P(\|X\| \geq t/2) + P(\|X'\| \geq t/2) = 2P(\|X\| \geq t/2)$, where X' is an independent copy of X such that $X^s = X - X'$. Note that $S_n^s = S_n - S'_n$.

Let M be a median of $\|S_n\|$. It is easy to see that

$$(6) \quad P(\|S_n\| - M \geq \lambda) \leq 2P(\|S_n^s\| \geq \lambda),$$

for all λ . (For, if $\|S_n\| - M \geq \lambda$, there is at least probability $1/2$ that $\|S'_n\| \leq M$ in which case $\|S_n - S'_n\| \geq \|S_n\| - \|S'_n\| \geq \|S_n\| - M \geq \lambda$.)

We now claim that in general in our present setting:

$$(7) \quad P(\|\tilde{S}_n\| \geq \varepsilon M) \geq \delta,$$

for absolute constants $\varepsilon, \delta \in (0, 1)$ to be determined later. (they will be determined in accordance with (12), (18), (20), (25) and (26), below). To prove (7), suppose that on the contrary we have:

$$(8) \quad P(\|\tilde{S}_n\| \geq \varepsilon M) \leq \delta.$$

Since the \tilde{X}_i are independent and identically distributed, by a maximal inequality for sums of independent and identically distributed random variables [7, Corollary 4] together with (8), we have:

$$(9) \quad P\left(\max_{1 \leq k \leq n} \|\tilde{S}_k\| \geq c_1 \varepsilon M\right) \leq c_1 P(\|\tilde{S}_n\| \geq \varepsilon M) \leq c_1 \delta,$$

where $c_1 \in [1, \infty)$ is an absolute constant. By the elementary inequality

$$P(\max_{1 \leq k \leq n} \|U_k\| \geq 2t) \leq P\left(\max_{1 \leq k \leq n} \left\| \sum_{i=1}^k U_i \right\| \geq t\right),$$

valid for all t if the U_i are independent (since if $\|U_k\| \geq 2t$ then $\|\sum_{i=1}^k U_i\| \geq t$ or $\|\sum_{i=1}^{k-1} U_i\| \geq t$), it follows from (9) that

$$(10) \quad P\left(\max_{1 \leq k \leq n} \|\tilde{X}_k\| \geq 2c_1 \varepsilon M\right) \leq c_1 \delta.$$

Let $L = 2c_1 \varepsilon M$. Set $Y_k = X_k \cdot 1_{\{\|X_k\| < L\}}$. Put $\tilde{Y}_k = \tilde{X}_k \cdot 1_{\{\|\tilde{X}_k\| < L\}}$. Note that Y_1, \dots, Y_n regularly cover \tilde{Y}_k for each k . Let $T_n = Y_1 + \dots + Y_n$ and put $\tilde{T}_n = \tilde{Y}_1 + \dots + \tilde{Y}_n$. By (10), we have:

$$(11) \quad P\left(\bigcup_{k=1}^n \{\tilde{X}_k \neq \tilde{Y}_k\}\right) \leq c_1 \delta.$$

Let $p = P(\|\tilde{X}_k\| \geq L)$. Note that this does not depend on k since the \tilde{X}_k are identically distributed. Note also that the left hand side of (11) is equal to $1 - (1 - p)^n$. Henceforth we will assume that

$$(12) \quad \delta < 1/(2c_1).$$

Now, if $x \in [0, 1]$ is such that $1 - (1 - x)^n \leq 1/2$, then $nx \leq 2(1 - (1 - x)^n)$. Then, using this observation, together with (11), (12) and the condition that X_1, \dots, X_n regularly cover \tilde{X}_1 :

$$(13) \quad \begin{aligned} P\left(\bigcup_{k=1}^n \{X_k \neq Y_k\}\right) &\leq \sum_{k=1}^n P(X_k \neq Y_k) \\ &= \sum_{k=1}^n P(\|X_k\| \geq L) \\ &= nP(\|\tilde{X}_1\| \geq L) = np \\ &\leq 2(1 - (1 - p))^n \\ &= 2P\left(\bigcup_{k=1}^n \{\tilde{X}_k \neq \tilde{Y}_k\}\right) \leq 2c_1 \delta. \end{aligned}$$

Now, by (5), (6) and (8), it follows that

$$P(\|S_n\| - M \geq 4\varepsilon M) \leq 32\delta.$$

Using (13), it then follows that:

$$(14) \quad P(\|T_n\| - M \geq 4\varepsilon M) \leq (32 + 2c_1)\delta.$$

Moreover, by (8) and (11):

$$(15) \quad P(\|\tilde{T}_n\| \geq \varepsilon M) \leq (1 + c_1)\delta.$$

Observe that $|\tilde{Y}_i| < L$ almost surely. Lemma 3 then shows that:

$$(16) \quad (E[\|\tilde{T}_n\|])^2 \geq c_2(E[\|\tilde{T}_n\|^2] - c_2^{-1}L^2),$$

where $c_2 \in (0, \infty)$ is an absolute constant.

Now, by (14) we have:

$$(17) \quad E[\|T_n\|^2] \geq [1 - (32 + 2c_1)\delta]M^2.$$

Henceforth, we will assume that δ is sufficiently small that

$$(18) \quad 1 - (32 + 2c_1)\delta \geq \frac{1}{2}.$$

Using Lemma 2 we see that $E[\|T_n\|^2] \leq 144E[\|\tilde{T}_n\|^2]$. Combining this with (17) and (18), we see that

$$(19) \quad E[\|\tilde{T}_n\|^2] \geq M^2/288.$$

Assume that $\varepsilon > 0$ is sufficiently small that $c_2^{-1}L^2 \leq M^2/(2 \cdot 288)$.

Since $L = 2c_1\varepsilon M$, this assumption is equivalent to:

$$(20) \quad \varepsilon \leq (48c_1)^{-1}c_2^{1/2}.$$

Thus by (19):

$$(21) \quad c_2^{-1}L^2 \leq E[\|\tilde{T}_n\|^2]/2.$$

Then, by (16),

$$(22) \quad (E[\|\tilde{T}_n\|])^2 \geq c_2(E[\|\tilde{T}_n\|^2] - c_2^{-1}L^2) \geq \frac{1}{2}c_2E[\|\tilde{T}_n\|^2].$$

The elementary inequality $P(|\Xi| \geq \lambda E[|\Xi|]) \geq (1 - \lambda)^2(E[|\Xi|])^2/E[|\Xi|^2]$ (see, e.g., [1, Exercise 3.3.11]) then implies that

$$(23) \quad P(\|\tilde{T}_n\| \geq \frac{1}{2}E[\|\tilde{T}_n\|]) \geq (1 - \frac{1}{2})^2 \cdot \frac{1}{2}c_2.$$

Now, by (19) and (22) we have $E[\|\tilde{T}_n\|] \geq (\frac{1}{2}c_2/288)^{1/2}M = c_2^{1/2}M/24$, so that (23) gives:

$$(24) \quad P(\|\tilde{T}_n\| \geq \frac{1}{2}c_2^{1/2}M/24) \geq c_2/8.$$

If we choose ε and δ such that

$$(25) \quad 0 < \varepsilon \leq c_2^{1/2}/48$$

and

$$(26) \quad 0 < (1 + c_1)\delta < c_2/8$$

and satisfying the other conditions required in the above argument (namely (12), (18) and (20)), we will obtain from (24) a contradiction to (15). Hence, if we take ε and δ to be absolute constants in $(0, 1)$ satisfying these assumptions, we obtain (7).

Now, combining (5) and (6), we see that:

$$(27) \quad P(\|S_n\| - M \geq \lambda) \leq 32P(\|\tilde{S}_n\| \geq \lambda/4),$$

for all λ . There are now two cases to be considered. Suppose first that $\lambda \leq 2M$. Then using (7):

$$(28) \quad P(\|S_n\| \geq \lambda) \leq 1 \leq \delta^{-1}P(\|\tilde{S}_n\| \geq \varepsilon M) \leq \delta^{-1}P(\|\tilde{S}_n\| \geq \varepsilon\lambda/2).$$

On the other hand, suppose that $\lambda > 2M$. In that case if $\|S_n\| \geq \lambda$ then $\|S_n\| - M > \lambda - \lambda/2 = \lambda/2$, so that

$$(29) \quad P(\|S_n\| \geq \lambda) \leq P(\|S_n\| - M \geq \lambda/2) \leq 32P(\|\tilde{S}_n\| \geq \lambda/4),$$

by (27). Inequality (1) follows from (28) for $\lambda \leq 2M$ and from (29) for $\lambda > 2M$, if we let $c = \max(32, 2/\varepsilon, \delta^{-1})$. \square

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